ALL SAINTS' COLLEGE

WA Exams Practice Paper E, 2016

Question/Answer Booklet

MATHEMATICS SPECIALIST UNIT 3

Section Two: Calculator-assumed

SOLUTIONS

Student Number:	In figures	
	In words	______________________
	Your name	______________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time for section: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	53	35
Section Two: Calculator-assumed	12	12	100	97	65
			Total	150	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Section Two: Calculator-assumed **65% (97 Marks)**

This section has **twelve (12)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 8 **(6 marks)**

Two complex numbers are given by $z = 4cis\left(\frac{\pi}{3}\right)$ and $w = \sqrt{3} - i$.

(a) Determine $\arg\left(\frac{z}{w}\right)$. (2 marks)

$$w = 2cis\left(-\frac{\pi}{6}\right)$$

$$\arg\frac{z}{w} = \frac{\pi}{3} - -\frac{\pi}{6}$$
$$= \frac{\pi}{2}$$

(b) Evaluate $\left| w \times \overline{w \times z} \right|$. (2 marks)

$$\left|w \times \overline{w \times z}\right| = \left|w \times \overline{w} \times \overline{z}\right|$$
$$= |w|^2 \times |\overline{z}|$$
$$= 4 \times 4$$
$$= 16$$

(c) Find the complex number u given that $\frac{z \times u}{20} = cis\left(\frac{3\pi}{4}\right)$. (2 marks)

$$u = \frac{20cis\frac{3\pi}{4}}{4cis\frac{\pi}{3}}$$
$$= 5cis\left(\frac{5\pi}{12}\right)$$

Question 9 **(8 marks)**

The positions of two particles, A and B, at time t seconds, $t \geq 0$, are given by the vector functions $\mathbf{r}_A = (2t+3)\mathbf{i} + (t^2+6)\mathbf{j}$ and $\mathbf{r}_B = (4t-3)\mathbf{i} + (5t)\mathbf{j}$.

(a) Determine the Cartesian equation for the path of particle A. (2 marks)

$$x = 2t+3 \;\Rightarrow\; t = \frac{x-3}{2}$$

$$y = t^2 + 6$$
$$= \left(\frac{x-3}{2}\right)^2 + 6$$
$$y = \frac{1}{4}x^2 - \frac{3}{2}x + \frac{21}{2}$$

(Last expansion optional)

(b) Determine the position of the particles at the instant that they collide. (3 marks)

$$i: \; 2t+3 = 4t-3 \;\Rightarrow\; t = 3$$
$$j: \; t^2 + 6 = 5t \;\Rightarrow\; t = 2, 3$$

Collide when $t = 3$

$$\mathbf{r}_A = \mathbf{r}_B = (2(3)+3)\mathbf{i} + (3^2+6)\mathbf{j}$$
$$= 9\mathbf{i} + 15\mathbf{j}$$

(c) Determine, with justification, which particle has the greatest speed just before they collide. (3 marks)

$$\mathbf{v}_A = (2)\mathbf{i} + (2t)\mathbf{j}$$
$$\left|\mathbf{v}_A\right|_{t=3} = \sqrt{2^2+6^2} = \sqrt{40}$$

$$\mathbf{v}_B = (4)\mathbf{i} + (5)\mathbf{j}$$
$$\left|\mathbf{v}_B\right|_{t=3} = \sqrt{4^2+5^2} = \sqrt{41}$$

B has greatest speed.

Question 10 **(5 marks)**

Given that $f(x) = x^4 + ax^3 + x^2 + bx - 170$ has factors $x-2$ and $x+5$, determine the real constants a and b and hence express $f(x)$ as the product of real factors.

$$f(2) = 8a + 2b - 150 = 0$$

$$f(-5) = -125a - 5b + 480 = 0$$

Solve simultaneously:

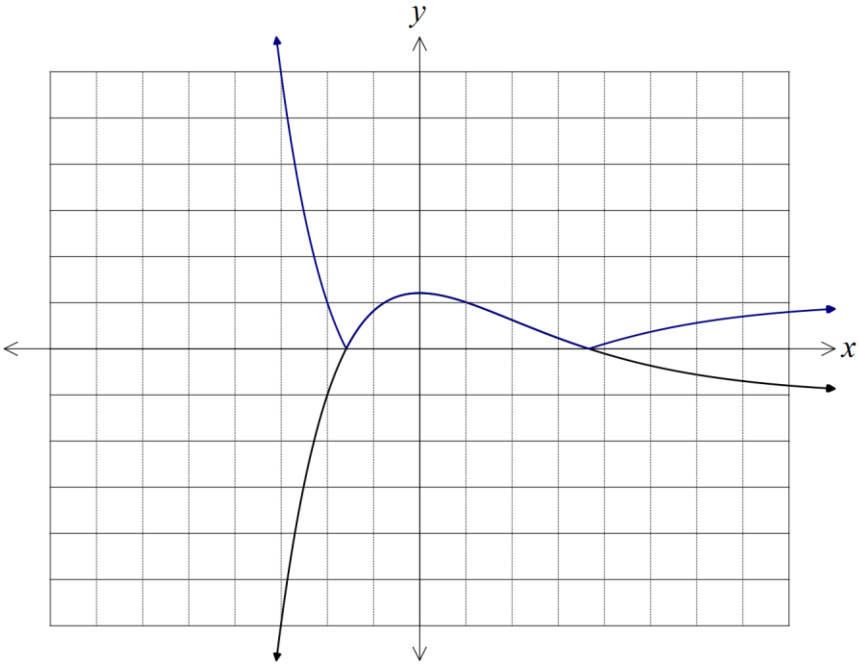
$$a = 1,\ b = 71$$

$$f(x) = x^4 + x^3 + x^2 + 71x - 170$$
$$f(x) = 0 \ \Rightarrow \ x = 2,\ \ -5,\ 1-4i,\ 1+4i$$

$$f(x) = (x-2)(x+5)(x-1+4i)(x-1-4i)$$
$$= (x-2)(x+5)(x^2 - 2x + 17)$$

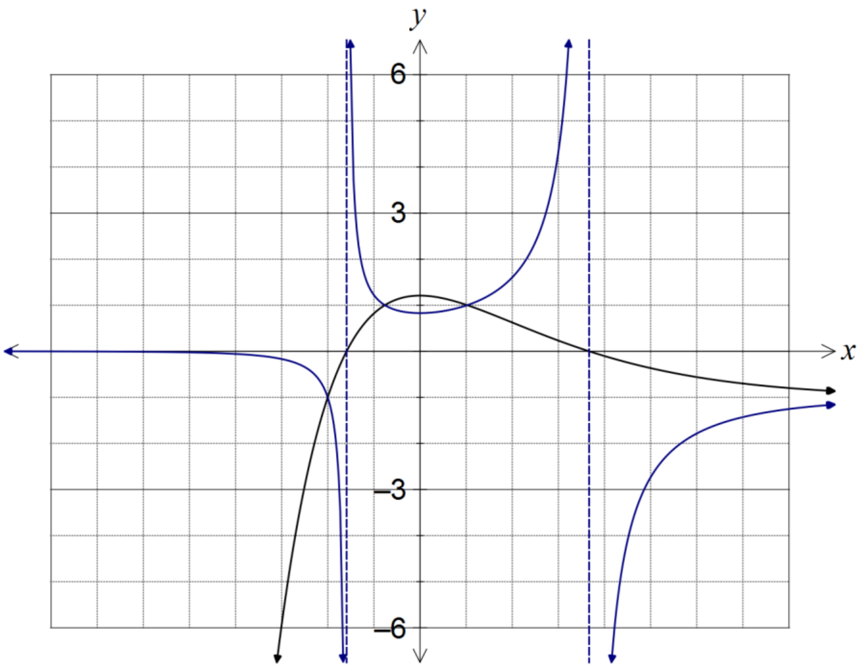
Question 11 **(7 marks)**

(a) The graph of $y = f(x)$ is shown below.
On the same axes, sketch the graph of $y = |f(x)|$. (2 marks)



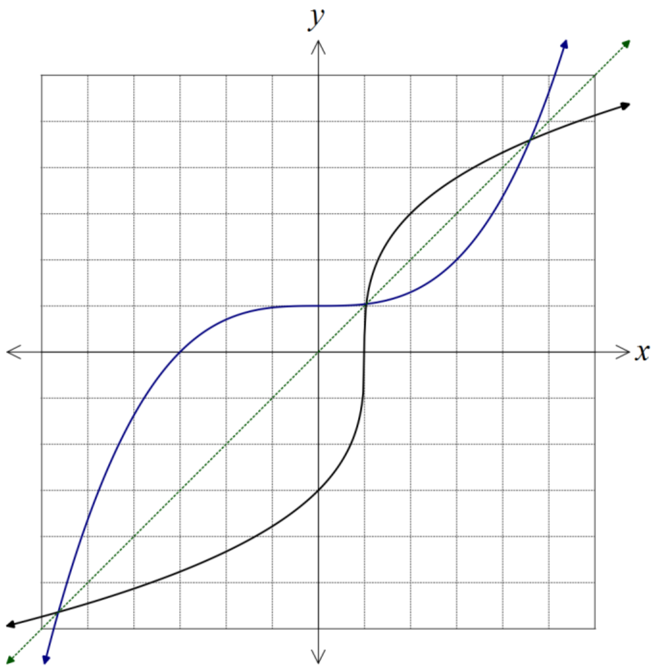
(b) The graph of $y = g(x)$ is shown below.

On the same axes, sketch the graph of $y = \dfrac{1}{g(x)}$, clearly indicating the position of any vertical asymptotes. (3 marks)



(c) The graph of $y = h(x)$ is shown below.

On the same axes, sketch the graph of the inverse of h, $y = h^{-1}(x)$. (2 marks)



Question 12 **(8 marks)**

Particle A leaves the point with position vector $-3\mathbf{i}-2\mathbf{j}$ and moves with constant velocity given by the vector $\mathbf{i}-2\mathbf{j}+\mathbf{k}$. At the same instant particle B leaves the point with position vector $\mathbf{i}-11\mathbf{j}+4\mathbf{k}$ and moves with constant velocity given by the vector $2\mathbf{i}-\mathbf{j}+2\mathbf{k}$.

All distances are in metres and times are in seconds.

(a) Determine the distance between the particles after one second. (3 marks)

$$\langle -3,-2,0\rangle + \langle 1,-2,1\rangle = \langle -2,-4,1\rangle$$
$$\langle 1,-11,4\rangle + \langle 2,-1,2\rangle = \langle 3,-12,6\rangle$$

$$\langle 3,-12,6\rangle - \langle -2,-4,1\rangle = \langle 7,-8,5\rangle$$

$$\left|\langle 7,-8,5\rangle\right| = \sqrt{114} \approx 10.7 \text{ m}$$

(b) Determine where the paths of A and B cross. (4 marks)

$$\langle -3,-2,0\rangle + t\langle 1,-2,1\rangle = \langle 1,-11,4\rangle + s\langle 2,-1,2\rangle$$

$$\begin{aligned} t-3 &= 2s+1 \\ -2-2t &= -11-s \\ t &= 4+2s \end{aligned}$$

$$t=\frac{14}{3},\ s=\frac{1}{3}$$

$$\langle 1,-11,4\rangle + \frac{1}{3}\langle 2,-1,2\rangle = \left\langle \frac{5}{3},\ -\frac{34}{3},\ \frac{14}{3}\right\rangle$$

(c) State, with reasoning, whether the particles collide. (1 mark)

No - times are different for A and B to be at point of intersection of paths.

Question 13 **(9 marks)**

A sphere of radius 7 has its centre at (2, -2, 1).

(a) Given that the point with coordinates (a, 1, -1) lies on the surface of the sphere, determine the value(s) of the constant a. (3 marks)

$$(x-3)^2+(y+3)^2+(z-2)^2=7^2$$

$$(a-2)^2+(1+2)^2+(-1-1)^2=7^2$$

$$a=-4,\ 8$$

Two points A and B have coordinates (19, 4, -1) and (-6,-11, 4) respectively.

(b) Determine the vector equation of the straight line through A and B. (2 marks)

$$\mathbf{BA}=\begin{bmatrix}19\\4\\-1\end{bmatrix}-\begin{bmatrix}-6\\-11\\4\end{bmatrix}=\begin{bmatrix}25\\15\\-5\end{bmatrix}=5\begin{bmatrix}5\\3\\-1\end{bmatrix}$$

$$\mathbf{r}=\begin{bmatrix}-6\\-11\\4\end{bmatrix}+t\begin{bmatrix}5\\3\\-1\end{bmatrix}$$

(c) Determine the coordinates of the points of intersection of the line through A and B and the sphere. (4 marks)

$$\left(5t-6-2\right)^2+\left(3t-11+2\right)^2+\left(4-t-1\right)^2=7^2$$

$$t=1,\ 3$$

$$\mathbf{r}_1=\begin{bmatrix}-6\\-11\\4\end{bmatrix}+\begin{bmatrix}5\\3\\-1\end{bmatrix}\Rightarrow(-1,\ -8,\ 3)$$

$$\mathbf{r}_1=\begin{bmatrix}-6\\-11\\4\end{bmatrix}+3\begin{bmatrix}5\\3\\-1\end{bmatrix}\Rightarrow(9,\ -2,\ 1)$$

Question 14 **(8 marks)**

(a) Use de Moivre's theorem to solve the equation $z^5 = 4 - 4i$, giving all solutions in exact polar form. (5 marks)

$$z^5 = 4\sqrt{2}cis\left(-\frac{\pi}{4}\right)$$

$$z = \sqrt{2}cis\left(-\frac{\pi}{4}\times\frac{1}{5}+\frac{2\pi n}{5}\right),\ n = \ldots,\ -1,\ 0,\ 1,\ 2,\ \ldots$$

$$z_1 = \sqrt{2}cis\left(\frac{3\pi}{4}\right)$$

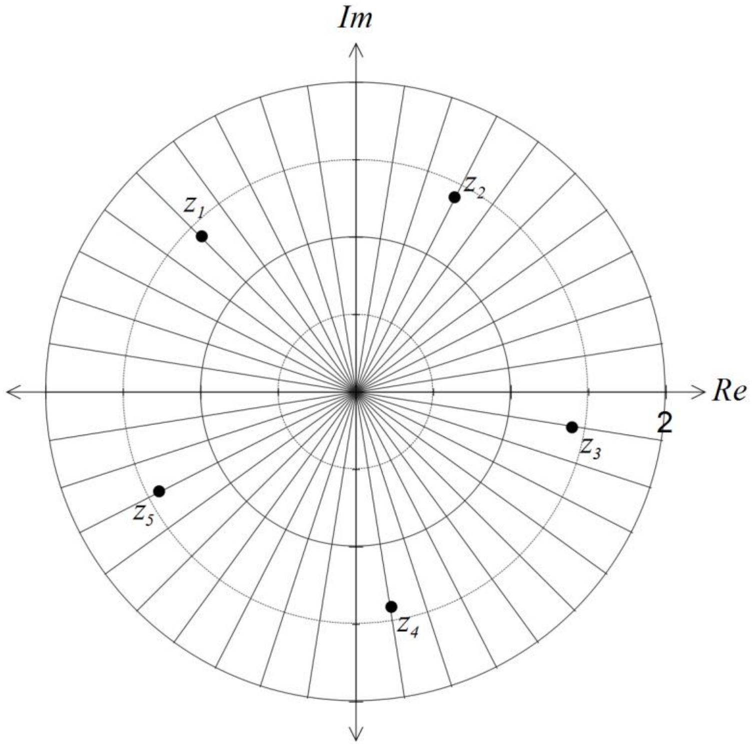
$$z_2 = \sqrt{2}cis\left(\frac{7\pi}{20}\right)$$

$$z_3 = \sqrt{2}cis\left(-\frac{\pi}{20}\right)$$

$$z_4 = \sqrt{2}cis\left(-\frac{9\pi}{20}\right)$$

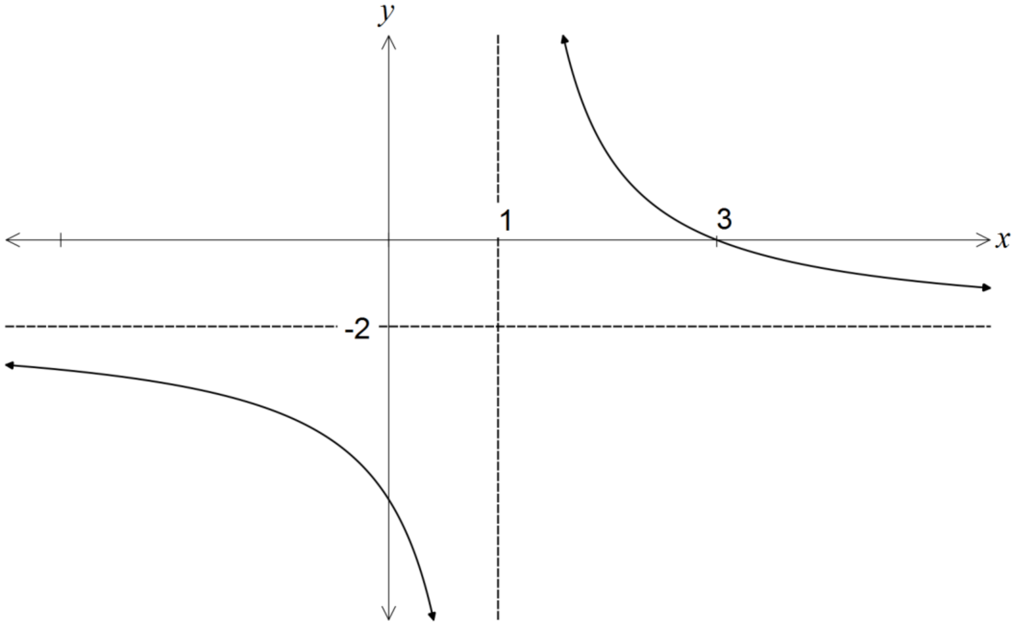
$$z_5 = \sqrt{2}cis\left(-\frac{17\pi}{20}\right)$$

(b) Show all solutions of the equation on the Argand diagram below. (3 marks)



Question 15 **(8 marks)**

The diagram below shows the graph of $y = \dfrac{ax+b}{cx+d}$, where a, b, c and d are non-zero constants.



(a) Express the values of b, c and d in terms of a. (6 marks)

Root: $a(3)+b=0 \;\Rightarrow\; b=-3a$

Horiz asymptote: $x \to \infty, y \to -2 \;\Rightarrow\; \dfrac{a}{c} = -2 \;\Rightarrow\; c = -\dfrac{1}{2}a$

Vert asymptote: $c(1)+d=0 \;\Rightarrow\; d=-c=\dfrac{1}{2}a$

(b) Determine the coordinates of the y-intercept of the graph. (2 marks)

$x=0 \;\Rightarrow\; y=\dfrac{b}{d}$

$y=\dfrac{-3a}{\frac{1}{2}a}=-6$

$(0,-6)$

Question 16 **(11 marks)**

A small object moves so that at any time *t* seconds, $t \geq 0$, its acceleration is given by

$\mathbf{a}(t) = \sin\left(\frac{t}{3}\right)\mathbf{i} - \cos\left(\frac{t}{3}\right)\mathbf{j}$.

At time $t = 3\pi$ the object is at the origin and has a velocity of $3\mathbf{i}$.

(a) Determine the velocity of the object at any time *t*. (3 marks)

$$\mathbf{v}(t) = \int \mathbf{a}(t)\ dt$$

$$= \left(-3\cos\left(\frac{t}{3}\right) + c_1\right)\mathbf{i} - \left(3\sin\left(\frac{t}{3}\right) + c_2\right)\mathbf{j}$$

$$\mathbf{v}(3\pi) = 3\mathbf{i} \ \Rightarrow\ c_1 = 0,\ c_2 = 0$$

$$\mathbf{v}(t) = -3\cos\left(\frac{t}{3}\right)\mathbf{i} - 3\sin\left(\frac{t}{3}\right)\mathbf{j}$$

(b) Show that the object moves with a constant speed. (2 marks)

$$s = |\mathbf{v}(t)|$$

$$= \sqrt{\left(-3\cos\left(\frac{t}{3}\right)\right)^2 + \left(-3\sin\left(\frac{t}{3}\right)\right)^2}$$

$$= 3 \text{ ms}^{-1}$$

Hence constant speed.

(c) Determine the initial position of the object. (3 marks)

$$\mathbf{r}(t) = \int \mathbf{v}(t)\ dt$$

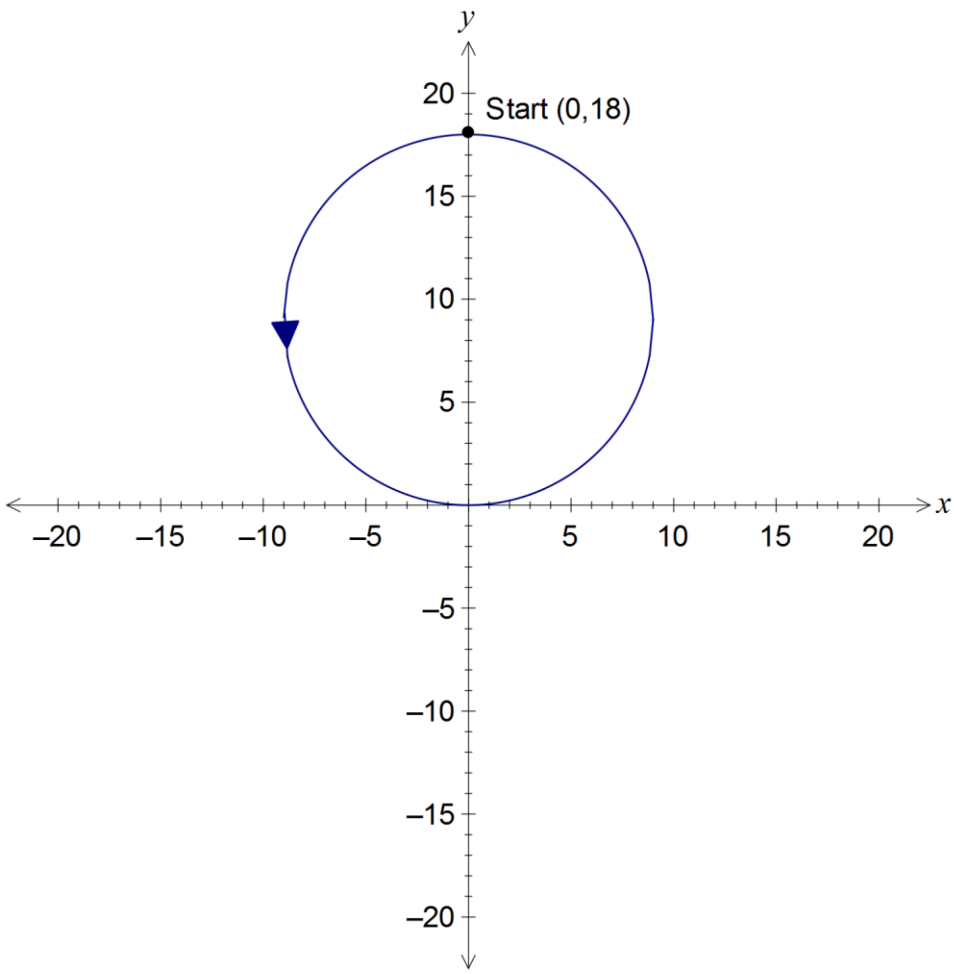
$$= \left(-9\sin\left(\frac{t}{3}\right) + c_1\right)\mathbf{i} + \left(9\cos\left(\frac{t}{3}\right) + c_2\right)\mathbf{j}$$

$$\mathbf{r}(3\pi) = \mathbf{0} \Rightarrow\ c_1 = 0,\ c_2 = 9$$

$$\mathbf{r}(t) = -9\sin\left(\frac{t}{3}\right)\mathbf{i} + \left(9\cos\left(\frac{t}{3}\right) + 9\right)\mathbf{j}$$

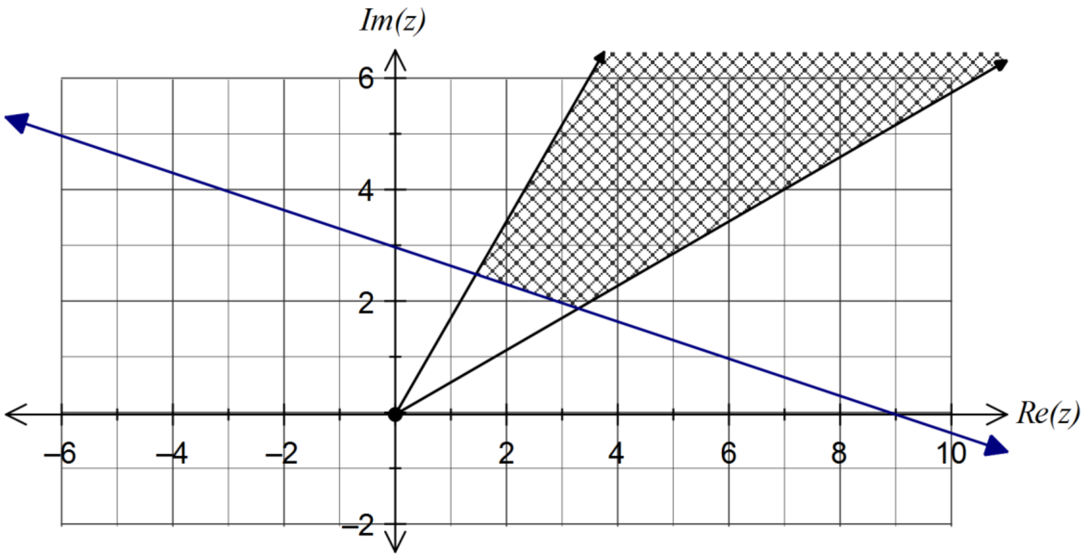
$$\mathbf{r}(0) = 18\mathbf{j}$$

(d) Sketch the path of the object on the axes below, indicating its initial position and direction of travel. (3 marks)



Question 17 **(10 marks)**

(a) The set of complex numbers that satisfy $\frac{\pi}{6} \le \arg(z) \le \frac{\pi}{3}$ is shown below.



(i) Add the set of complex numbers $|z+1| \ge |z-1-6i|$ to the diagram and clearly shade the region that satisfies both inequalities. (3 marks)

(ii) Determine, in exact form, the minimum value of $\mathrm{Im}(z)$ in the region that satisfies both inequalities. (3 marks)

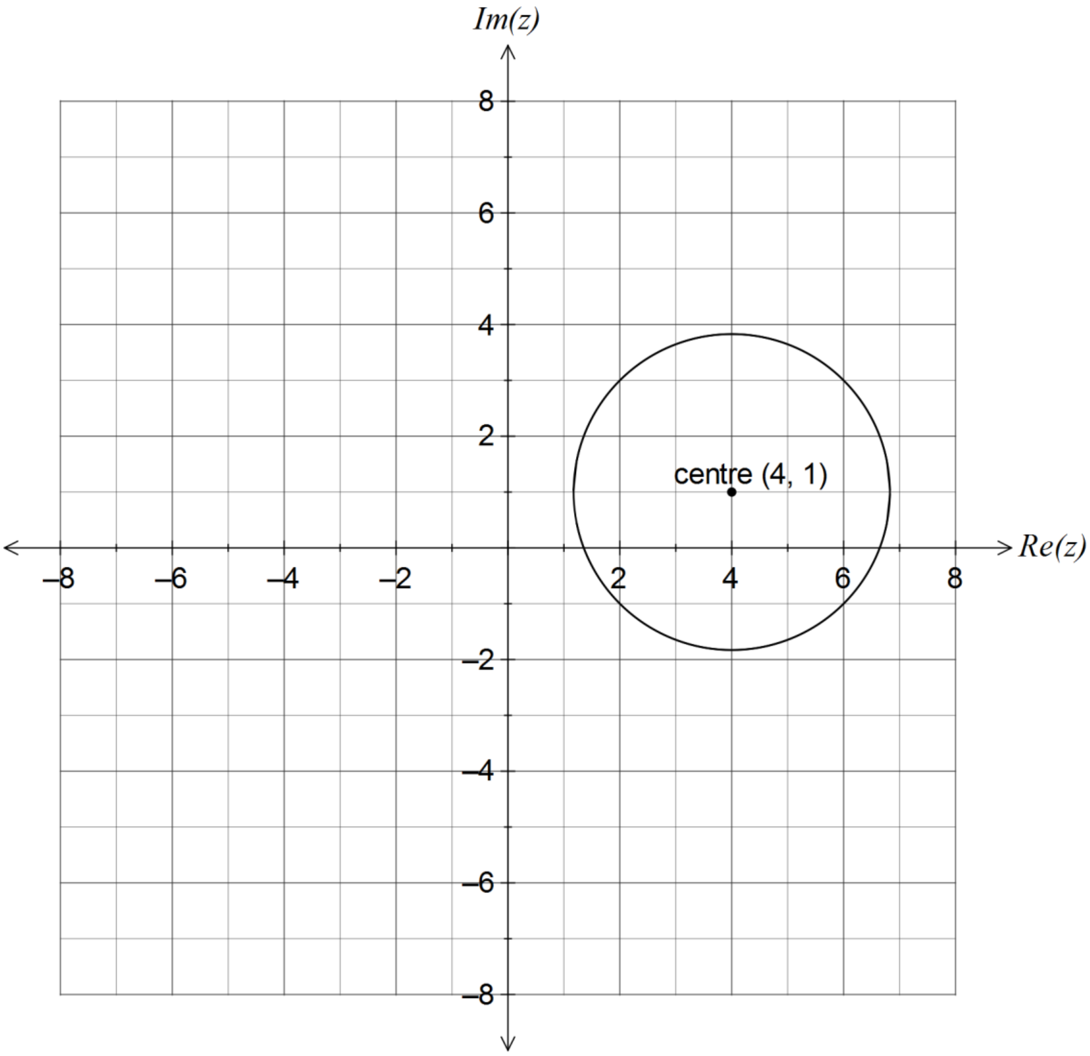
$(A)\ y = \tan\left(\frac{\pi}{6}\right)x \Rightarrow x = \sqrt{3}y$

$(B)\ y = 3 - \frac{1}{3}x \Rightarrow x = 9 - 3y$

Min Im(z) is y-coord at intersection of (A) and (B)

$$\sqrt{3}y = 9 - 3y \Rightarrow y = \frac{9}{3+\sqrt{3}} = \frac{3\left(3-\sqrt{3}\right)}{2}$$

(b) Sketch region satisfying the equation $|z+3i| = 2|z-3|$ on the Argand diagram below. (4 marks)



$2|x-3+iy| = |x+i(y+3)|$

$4x^2+4y^2-24x+36 = x^2+y^2+6y+9$

$x^2+y^2-8x-2y+27=0$

$(x-4)^2+(y-1)^2=8$

Circle centre (4, 1) radius $\sqrt{8}$

Question 18 **(10 marks)**

The equations of three planes are $x+3y-z=-6$, $3x-y+2z=7$ and $2x-2y+az=b$, where a and b are real constants.

(a) Determine the values of a and b so that the three planes intersect in three parallel lines. (2 marks)

Elimination steps will reduce system to $0x+0y+0z=k,\ k\neq 0$.

Using CAS:

$$\begin{bmatrix}1 & 3 & -1 & -6\\ 3 & -1 & 2 & 7\\ 2 & -2 & a & b\end{bmatrix}$$

$$\begin{bmatrix}1 & 3 & -1 & -6\\ 3 & -1 & 2 & 7\\ 2 & -2 & a & b\end{bmatrix}$$

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$$\begin{bmatrix}1 & 3 & -1 & -6\\ 0 & 1 & -\frac{1}{2} & -\frac{5}{2}\\ 0 & 0 & 1 & \frac{b-8}{a-2}\end{bmatrix}$$

Hence require that $a-2=0 \Rightarrow a=2$ and $b\neq 8$.

(b) When $a=1$ the three planes intersect at the point with coordinates (p, q, 7). Determine the values of b, p and q. (3 marks)

Using CAS:

$$\begin{bmatrix}1 & 3 & -1 & -6\\ 3 & -1 & 2 & 7\\ 2 & -2 & a & b\end{bmatrix}|a=1$$

$$\begin{bmatrix}1 & 3 & -1 & -6\\ 3 & -1 & 2 & 7\\ 2 & -2 & 1 & b\end{bmatrix}$$

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$$\begin{bmatrix}1 & 0 & 0 & \frac{b-5}{2}\\ 0 & 1 & 0 & \frac{-(b-3)}{2}\\ 0 & 0 & 1 & -b+8\end{bmatrix}$$

Require that $-b+8=7 \Rightarrow b=1$.

$$\begin{bmatrix}1 & 3 & -1 & -6\\ 3 & -1 & 2 & 7\\ 2 & -2 & a & b\end{bmatrix}|a=1|b=1$$

$$\begin{bmatrix}1 & 3 & -1 & -6\\ 3 & -1 & 2 & 7\\ 2 & -2 & 1 & 1\end{bmatrix}$$

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$$\begin{bmatrix}1 & 0 & 0 & -2\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 7\end{bmatrix}$$

$p=-2,\ q=1$.

(c) Given that the three planes intersect in one straight line, determine the equation of this line. (5 marks)

Elimination steps will reduce system to $0x+0y+0z=0$.
Hence, from (a), we see that $a=2$ and $b=8$.

From second line of reduced echelon form we see that
$y-\frac{1}{2}z=-\frac{5}{2} \;\Rightarrow\; z=2y+5$.

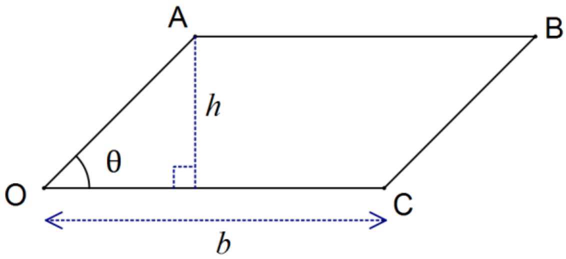
Substitute into first line of ref form to get
$x+3y-(2y+5)=-6 \;\Rightarrow\; x=-y-1$.

Parametric equation of line: $x=-y-1,\; y=y,\; z=2y+5$.

Vector form: $\mathbf{r}=\begin{bmatrix}-1\\0\\5\end{bmatrix}+\lambda\begin{bmatrix}-1\\1\\2\end{bmatrix}$.

Question 19 **(7 marks)**

Consider the parallelogram *OABC* shown below, where $\mathbf{OA} = \mathbf{a}$, $\mathbf{OC} = \mathbf{c}$ and the angle between $\mathbf{a}$ and $\mathbf{c}$ is θ.



(a) Use the geometric definition of the cross product to show that the area of the parallelogram is $|\ \mathbf{a} \times \mathbf{c}\ |$. (3 marks)

$$\mathbf{a} \times \mathbf{c} = |\mathbf{a}| \cdot |\mathbf{c}| \cdot \sin\theta \cdot \hat{\mathbf{n}}$$

Since $\hat{\mathbf{n}}$ is a unit vector, then

$$|\ \mathbf{a} \times \mathbf{c}\ | = |\mathbf{a}| \cdot |\mathbf{c}| \cdot \sin\theta \cdot 1$$

$$= (|\mathbf{c}|) \cdot (|\mathbf{a}| \cdot \sin\theta)$$

$$= b \times h$$

$=$ Area of parallelogram

(b) Let *O* be the origin and the coordinates of *A* and *B* be (1, 2, 0) and (3, 1, *k*) respectively, where *k* is a constant. Determine the value(s) of *k* given that the area of the parallelogram is $3\sqrt{5}$.

(4 marks)

$$\mathbf{c} = \mathbf{b} - \mathbf{a}$$

$$= \langle 3,1,k \rangle - \langle 1,2,0 \rangle$$

$$= \langle 2,-1,k \rangle$$

$$\langle 1,2,0 \rangle \times \langle 2,-1,k \rangle = \langle 2k,-k,-5 \rangle$$

$$|\langle 2k,-k,-5 \rangle| = 3\sqrt{5}$$

$$5(k^2+5) = \left(3\sqrt{5}\right)^2$$

$$k = \pm 2$$

End of questions

Additional working space

Question number: _________